1 Mathematical Formulation

I prefer to use

$$\exp\{\mathrm{i}\,(\omega\,t - \beta\,z)\}$$

for the waves in z direction. Since all equations will be in the frequency domain, in the following text $\exp(\mathrm{i}\,\omega\,t)$ will be omitted, and the symbol t will be re-used for the function parameter, instead of time. In addition, $\epsilon = \epsilon_0$ and $\mu = \mu_0$ for convenience.

From the two Maxwell equations inside the waveguide

$$\nabla \times \boldsymbol{H} = \mathrm{i}\,\omega\,\epsilon\,\boldsymbol{E} \tag{1a}$$

$$\nabla \times \boldsymbol{E} = -\mathrm{i}\,\omega\,\mu\,\boldsymbol{H} \tag{1b}$$

One write the Helmholtz equation using $\boldsymbol{U} \in \{\boldsymbol{E}, \boldsymbol{H}\}$

$$\nabla^2 \boldsymbol{U} + \omega^2\,\mu\,\epsilon\,\boldsymbol{U} = 0 \tag{2}$$

Consider first only the ideal electric conductor ($\sigma \to \infty$) for waveguide walls

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \tag{3a}$$

$$\boldsymbol{n} \cdot \boldsymbol{B} = 0 \tag{3b}$$

Let $z = 0$ be the reference plane. (x', y') is a point in this plane, the point has the azimuthal angle

$$\xi := \arg(x', y') \tag{4}$$

The "trajectory" of a point in the helical waveguide is a helix at radius

$$r := \sqrt{x'^2 + y'^2} \tag{5}$$

A given tuple (x', y') or (r, ξ) labels one of the helical trajectories in Cartesian or cylindrical coordinates, respectively.

A point at $(x', y', 0)$ will follow the helical trajectory to an arbitrary z. This process is a counterclockwise affine rotation by angle $\iota\, z$, where the constant ι is the pitch of the helix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(\iota\, z) & -\sin(\iota\, z) & 0 \\ \sin(\iota\, z) & \cos(\iota\, z) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= \mathsf{M}(z)} \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \tag{6a}$$

The rotation matrix is denoted as M. To obtain the label of helix, the inverse transformation can be performed

$$\begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(\iota\, z) & \sin(\iota\, z) & 0 \\ -\sin(\iota\, z) & \cos(\iota\, z) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{= \mathsf{M}^{-1}(z) \,=\, \mathsf{M}(-z)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -z \end{pmatrix} \tag{6b}$$

Hence,

$$\frac{\partial x'}{\partial x} = \cos(\iota\, z) \quad \text{and} \quad \frac{\partial x'}{\partial y} = \sin(\iota\, z) \tag{7a}$$

$$\frac{\partial y'}{\partial x} = -\sin(\iota\, z) \quad \text{and} \quad \frac{\partial y'}{\partial y} = \cos(\iota\, z) \tag{7b}$$

and the second derivatives of these are zero.

The key for a wave propagating (or whatever) in a *rotating* z direction is the Ansatz of variable pseudo-separation

$$\boldsymbol{U}(x, y, z) = \mathsf{M}(z)\, \boldsymbol{u}(x', y')\, \exp(-\mathrm{i}\, \beta\, z) \tag{8}$$

Note that the transformation for z component is identity. For an easier interpretation, one can read

$$\boldsymbol{u}(x', y') = \boldsymbol{U}(x', y', 0)$$

Since the transformation $\mathsf{M}(z)$ and the exp(...) term is independent of x and y

$$\frac{\partial^2 \boldsymbol{U}}{\partial x^2} + \frac{\partial^2 \boldsymbol{U}}{\partial y^2} = \mathsf{M}(z) \left(\frac{\partial^2 \boldsymbol{u}}{\partial x^2} + \frac{\partial^2 \boldsymbol{u}}{\partial y^2} \right) \exp(-\mathrm{i}\, \beta\, z)$$

The derivative of z is

$$\frac{\partial \boldsymbol{U}}{\partial z} = \left\{ \frac{\partial M}{\partial z}\, \boldsymbol{u} + M\, \frac{\partial \boldsymbol{u}}{\partial z} - \mathrm{i}\, \beta\, M\, \boldsymbol{u} \right\} \exp(-\mathrm{i}\, \beta\, z)$$

and the second-order derivative is

$$\begin{aligned} \frac{\partial^2 \boldsymbol{U}}{\partial z^2} &= \left\{ \frac{\partial^2 M}{\partial z^2}\, \boldsymbol{u} + 2\, \frac{\partial M}{\partial z} \left(\frac{\partial \boldsymbol{u}}{\partial z} - \mathrm{i}\, \beta\, \boldsymbol{u} \right) \right. \\ &\left. + M \left(\frac{\partial^2 \boldsymbol{u}}{\partial z^2} - 2\, \mathrm{i}\, \beta\, \frac{\partial \boldsymbol{u}}{\partial z} - \beta^2\, \boldsymbol{u} \right) \right\} \exp(-\mathrm{i}\, \beta\, z) \end{aligned}$$

$$\begin{aligned} \nabla^2 \boldsymbol{U} = \frac{\partial^2 \boldsymbol{U}}{\partial x^2} + \frac{\partial^2 \boldsymbol{U}}{\partial y^2} + \frac{\partial^2 \boldsymbol{U}}{\partial z^2} &= \left\{ \frac{\partial^2 M}{\partial z^2}\, \boldsymbol{u} + 2\, \frac{\partial M}{\partial z} \left(\frac{\partial \boldsymbol{u}}{\partial z} - \mathrm{i}\, \beta\, \boldsymbol{u} \right) \right. \\ &\left. + M \left(\nabla^2 \boldsymbol{u} - 2\, \mathrm{i}\, \beta\, \frac{\partial \boldsymbol{u}}{\partial z} - \beta^2\, \boldsymbol{u} \right) \right\} \exp(-\mathrm{i}\, \beta\, z) \end{aligned} \tag{9}$$

Treat the Helmholtz equation in eq. (2) as

$$M\, M^{-1}\, \nabla^2 \boldsymbol{U} + \omega^2\, \mu\, \epsilon\, \boldsymbol{U} = 0 \tag{10}$$

then the transformation M ahead of each term and the $\exp(-\mathrm{i}\, \beta\, z)$ part can be cancelled

$$\begin{aligned} M^{-1}\, \frac{\partial^2 M}{\partial z^2}\, \boldsymbol{u} + 2\, M^{-1}\, \frac{\partial M}{\partial z} \left(\frac{\partial \boldsymbol{u}}{\partial z} - \mathrm{i}\, \beta\, \boldsymbol{u} \right) \\ + \nabla^2 \boldsymbol{u} - 2\, \mathrm{i}\, \beta\, \frac{\partial \boldsymbol{u}}{\partial z} + (\omega^2\, \mu\, \epsilon - \beta^2)\, \boldsymbol{u} = 0 \end{aligned} \tag{11}$$

It can be found that

$$M^{-1}\, \frac{\partial^2 M}{\partial z^2} = \begin{pmatrix} -\iota^2 & 0 & 0 \\ 0 & -\iota^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12a}$$

and

$$M^{-1}\, \frac{\partial M}{\partial z} = \begin{pmatrix} 0 & -\iota & 0 \\ \iota & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12b}$$

Note that $\iota = 0$ should in agreement with the usual variable separation in a non-rotating coordinates. (Hint: the terms with M vanish. Besides, x' and y' are in this case independent of z, thus $\partial \boldsymbol{u}/\partial z = 0$.)

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{u}}{\partial x^2} &= \cos(\iota\, z)\,\frac{\partial}{x}\frac{\partial \boldsymbol{u}}{\partial x'} - \sin(\iota\, z)\,\frac{\partial}{x}\frac{\partial \boldsymbol{u}}{\partial y'} \\
&= \cos(\iota\, z)\left(\cos(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'^2} - \sin(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'\,\partial y'}\right) \\
&\quad - \sin(\iota\, z)\left(\cos(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'\,\partial y'} - \sin(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial y'^2}\right) \\
&= \cos^2(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'^2} + \sin^2(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial y'^2} - 2\,\sin(\iota\, z)\,\cos(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'\,\partial y'}
\end{aligned}$$

For the twice derivative of y, since there is no transformation of triangle functions in the last steps, it is just to replace whereever cos by $-\sin$ while sin by cos:

$$\frac{\partial^2 \boldsymbol{u}}{\partial y^2} = \sin^2(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'^2} + \cos^2(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial y'^2} + 2\,\sin(\iota\, z)\,\cos(\iota\, z)\,\frac{\partial^2 \boldsymbol{u}}{\partial x'\,\partial y'}$$

Therefore,

$$\nabla_\perp^2 \boldsymbol{u} = \frac{\partial^2 \boldsymbol{u}}{\partial x^2} + \frac{\partial^2 \boldsymbol{u}}{\partial y^2} = \frac{\partial^2 \boldsymbol{u}}{\partial x'^2} + \frac{\partial^2 \boldsymbol{u}}{\partial y'^2} = \nabla_\perp'^2 \boldsymbol{u}$$

The calculation of z derivatives is tricky. First, from eq. (6b)

$$\frac{\partial}{\partial z}\begin{pmatrix} x' \\ y' \end{pmatrix} = \iota \begin{pmatrix} -\sin(\iota\, z) & \cos(\iota\, z) \\ -\cos(\iota\, z) & -\sin(\iota\, z) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

substituting (x, y) by eq. (6a)

$$\frac{\partial}{\partial z}\begin{pmatrix} x' \\ y' \end{pmatrix} = \iota \begin{pmatrix} -\sin(\iota\, z) & \cos(\iota\, z) \\ -\cos(\iota\, z) & -\sin(\iota\, z) \end{pmatrix}\begin{pmatrix} \cos(\iota\, z) & -\sin(\iota\, z) \\ \sin(\iota\, z) & \cos(\iota\, z) \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

that is

$$\frac{\partial}{\partial z}\begin{pmatrix} x' \\ y' \end{pmatrix} = \iota \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \iota\, y' \\ -\iota\, x' \end{pmatrix} \tag{13}$$

with this shortcut, $\partial \boldsymbol{u}/\partial z$ can be write as

$$\frac{\partial \boldsymbol{u}(x',y')}{\partial z} = \frac{\partial \boldsymbol{u}}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial \boldsymbol{u}}{\partial y'}\frac{\partial y'}{\partial z} = \iota\,\left(y'\,\frac{\partial \boldsymbol{u}}{\partial x'} - x'\,\frac{\partial \boldsymbol{u}}{\partial y'}\right) \tag{14a}$$

The second-order derivative is

$$\frac{\partial^2 \boldsymbol{u}(x',y')}{\partial z^2} = \iota^2\,\left[y'^2\,\frac{\partial^2 \boldsymbol{u}}{\partial x'^2} + x'^2\,\frac{\partial^2 \boldsymbol{u}}{\partial y'^2} - \left(x'\,\frac{\partial \boldsymbol{u}}{\partial x'} + y'\,\frac{\partial \boldsymbol{u}}{\partial y'} + 2\,x'\,y'\,\frac{\partial^2 \boldsymbol{u}}{\partial x'\,\partial y'}\right)\right] \tag{14b}$$

Equation (10) can be written in (x',y') coordinates

$$\begin{aligned}
-\iota^2\begin{pmatrix}u_x\\u_y\\0\end{pmatrix} + 2\,\iota\,y'\,\frac{\partial}{\partial x'}\begin{pmatrix}-u_y\\u_x\\0\end{pmatrix} + 2\,\iota\,x'\,\frac{\partial}{\partial y'}\begin{pmatrix}u_y\\-u_x\\0\end{pmatrix} + 2\,\mathrm{i}\,\beta\begin{pmatrix}u_y\\-u_x\\0\end{pmatrix}&\\
+(1+\iota^2\,y'^2)\,\frac{\partial^2}{\partial x'^2}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} + (1+\iota^2\,x'^2)\,\frac{\partial^2}{\partial y'^2}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix}&\\
-\iota^2\,x'\,\frac{\partial}{\partial x'}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} - \iota^2\,y'\,\frac{\partial}{\partial y'}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} - 2\,\iota^2\,x'\,y'\,\frac{\partial^2}{\partial x'\,\partial y'}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix}&\\
-2\,\mathrm{i}\,\beta\,\iota\,y'\,\frac{\partial}{\partial x'}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} + 2\,\mathrm{i}\,\beta\,\iota\,x'\,\frac{\partial}{\partial y'}\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} + (\omega^2\,\mu\,\epsilon - \beta^2)\,\begin{pmatrix}u_x\\u_y\\u_z\end{pmatrix} &= 0
\end{aligned}$$